Density of Euclid's orchard

Let $E = \{(x, y) \in \mathbb{P}^2 : \gcd(x, y) = 1\}$ be Euclid's orchard. For $n \in \mathbb{P}$ write $\mathbf{n} = \{1, \ldots, n\}$, and

$$E_n = E \cap \mathbf{n}^2 = \{(x, y) \in \mathbf{n}^2 : \gcd(x, y) = 1\}.$$

Also write

$$E_n^1 = \{(x, y) \in E_n : x \leq y\} \qquad \text{and} \qquad E_n^1 = \{(x, y) \in E_n : x \geq y\}.$$

Clearly $|E_n^1| = |E_n^2|$, $E_n = E_n^1 \cup E_n^2$ and $E_n^1 \cap E_n^2 = \{(1, 1)\}$. Thus, $|E_n| = 2|E_n^1| - 1$.

For $1 \leq m \leq n$, let

$$E_n^1(m) = \{(x, y) \in E_n^1 : y = m\} = \{(x, m) : x \in \mathbf{m},\ \gcd(x, m) = 1\}.$$

So $E_n^1 = E_n^1(1) \sqcup \cdots \sqcup E_n^1(n)$. By definition, $|E_n^1(m)| = \varphi(m)$, so

$$|E_n^1| = \varphi(1) + \cdots + \varphi(n).$$

Denote this sum by $\Phi(n)$. So $|E_n| = 2\Phi(n) - 1$.

The numbers $\Phi(n)$ are on [1, A002088], which states that

$$\Phi(n) = \frac{3n^2}{\pi^2} + O(n \log n).$$

It follows that

$$|E_n| = 2\Phi(n) - 1 = \frac{6n^2}{\pi^2} + O(n \log n),$$

and the density of E_n in $\mathbf{n}^2$ is

$$\frac{|E_n|}{n^2} = \frac{6}{\pi^2} + O(\log n / n),$$

As $n \to \infty$, therefore $\frac{|E_n|}{n^2} \to \frac{6}{\pi^2} \approx 0.60792710185$.

References

[1] The Online Encyclopedia of Integer Sequences. Published electronically at oeis.org, 2018.